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1) a) $x = y^2 + 4z^2$.

The traces in $\underline{x = k}$ are $y^2 + 4z^2 = k$

$k > 0$: Get a family of ellipses.

$k = 0$: point at the origin

$k < 0$: empty

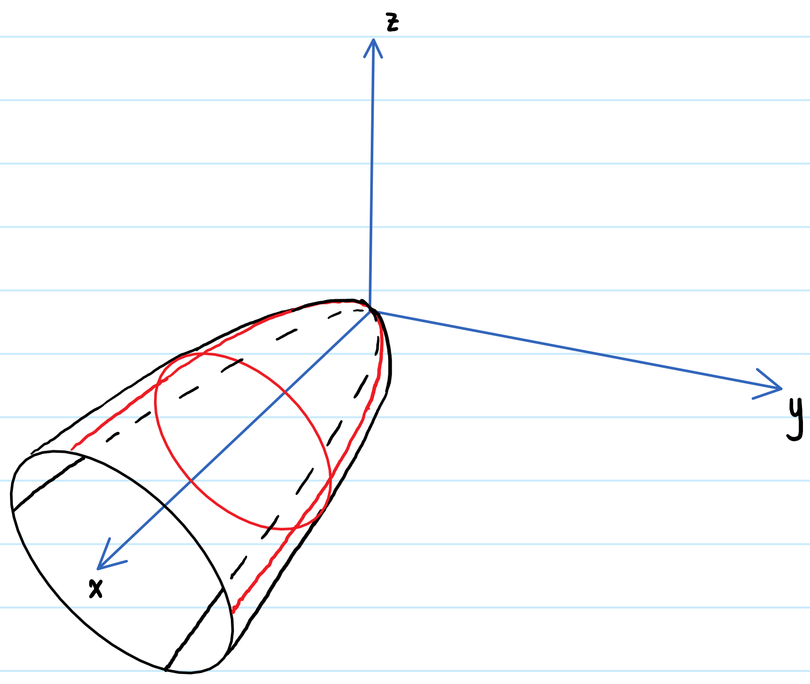
The traces in $y = k$ are $x = 4z^2 + k^2$

So we have a family of parabolas opening in the positive x-direction.

The traces in $z = k$ are $x = y^2 + 4k^2$

which is again a family of parabolas opening in the positive x-direction.

- So finally, we see the graph is what is called an <u>elliptic paraboloid</u> w/ axis the x-axis and the vertex origin.



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1b) $16x^2 = y^2 + 4z^2$

The traces in $\underline{x = k}$ are $y^2 + 4z^2 = 16k^2$

- $\underline{k \neq 0}$: We have a family of ellipses

  $\underline{k = 0}$: point at the origin.

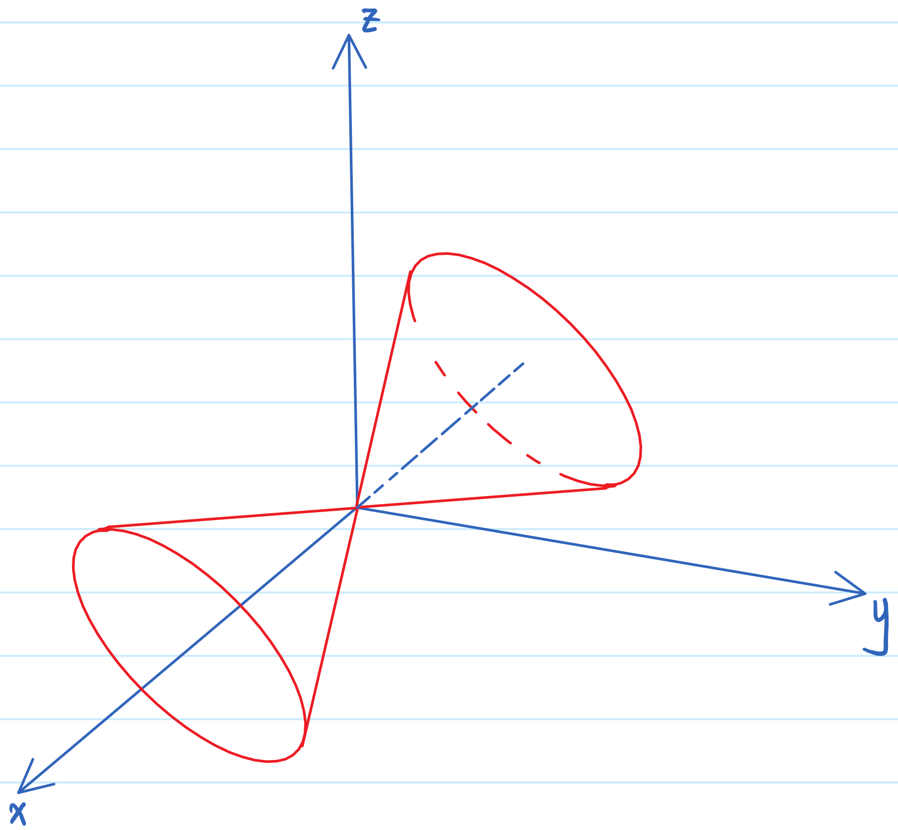
The traces for $\underline{y = k}$ are $16x^2 - 4z^2 = k^2$

- If $k \neq 0$, we get hyperbolas.
- If $k = 0$, we get two intersecting lines

The traces for $\underline{z = k}$ are $16x^2 - y^2 = 4k^2$

- $k \neq 0$, we get hyperbolas
- $k = 0$, we get two intersecting lines.

- So we get an elliptic cone with axis x-axis and vertex $(0,0,0)$



2)

a) $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2+1-1} \cdot \left(\sqrt{x^2+y^2+1}+1\right)$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1} + 1 = \sqrt{1} + 1 = 2$$

b) $$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$

Ans Let $f(x,y) = \dfrac{xy^4}{x^2+y^8}$

Let's approach $(0,0)$ along the y-axis i.e $x = 0$.

Then,

$$f(0,y) = \frac{0}{y^8} = 0 \quad \text{for all } y \neq 0.$$

Therefore,

$f(x,y) \longrightarrow (0,0)$ as $(x,y) \to (0,0)$ along the y-axis.

On the other hand, let $(x,y) \to (0,0)$ along the parabola $x = y^4$, we have

$$f(x,y) = f(y^2.y) = \frac{y^4.y^4}{(y^4)^2+y^8} = \frac{y^8}{2y^8} = \frac{1}{2} \quad \text{for all } y \neq 0.$$

Therefore, $f(x,y) \longrightarrow \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x = y^2$

Since f has two different limits along two different lines, $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^2+y^8}$ D.N.E.

3) a) $w = \ln(x+2y+3z)$

$$\frac{\partial w}{\partial x} = \frac{1}{x+2y+3z}, \quad \frac{\partial w}{\partial y} = \frac{2}{x+2y+3z}, \quad \frac{\partial w}{\partial z} = \frac{3}{x+2y+3z}$$

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b) $\phi(x,y,z,t) = \dfrac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

$\dfrac{\partial \phi}{\partial x} = \dfrac{\alpha}{\gamma z + \delta t^2}$, $\dfrac{\partial \phi}{\partial y} = \dfrac{2\beta y}{\gamma z + \delta t^2}$, $\dfrac{\partial \phi}{\partial z} = \dfrac{\alpha x + \beta y^2}{(\gamma z + \delta t^2)^2} \cdot -\gamma$, $\dfrac{\partial \phi}{\partial t} = \dfrac{\alpha x + \beta y^2}{(\gamma t + \delta t^2)^2} \cdot -2\delta t$

4) $f(x,y) = xe^{xy}$

$f_x = e^{xy} + xye^{xy}$; $f_y = x^2 e^{xy}$

$f_x(2,0) = e^{2\cdot 0} + 0e^{2\cdot 0} = 1$
$f_y(2,0) = 4e^0 = 4$.

Then equation of the tangent plane to $z = xe^{xy}$ at $(2,0,2)$ is

$z - 2 = 1(x-2) + 4(y-0)$
$\Rightarrow z - 2 = x - 2 + 4y \Rightarrow z = x + 4y$

5) The solid lies inside the cylinder $x^2+y^2=4$
and between surfaces $z = \sqrt{64-4(x^2+y^2)}$ and $z = -\sqrt{64-4(x^2+y^2)}$

Therefore, $V = \displaystyle\iint_{x^2+y^2\le 4} \left[\sqrt{64-4(x^2+y^2)}\ (-\sqrt{64-4(x^2+y^2)}\ )\right] dA = \iint_{x^2+y^2\le 4} 2\sqrt{64-4(x^2+y^2)}\ dA$

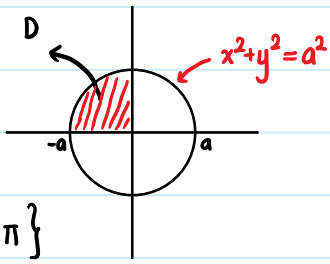
Now converting to polar coordinates, we get

$V = \displaystyle\int_0^{2\pi}\int_0^2 2\sqrt{64-4r^2}\ r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_0^2 r\sqrt{64-4r^2}\,dr$ Let $u = 64 - 4r^2$, $r = 0$, $u = 64$
$du = -8r\,dr$, $r = 2$, $u = 48$

$= 4\pi \cdot \left(-\dfrac{1}{8}\right) \displaystyle\int_{64}^{48} \sqrt{u}\,du$

$= -\dfrac{\pi}{2}\left[\dfrac{u^{3/2}}{3/2}\right]_{64}^{48} = -\dfrac{\pi}{3}\left[48\sqrt{48} - 64\sqrt{64}\right] = -\dfrac{\pi}{3}\left[48\cdot 4\sqrt{3} - 64\cdot 8\right] = \dfrac{64\pi}{3}\left[8 - 3\sqrt{3}\right]$

6) $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y \, dx \, dy$

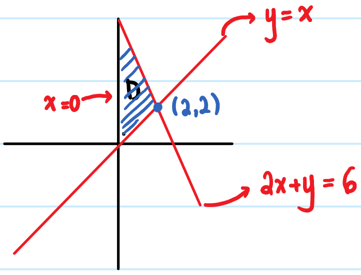


$D = \{(r,\theta) \mid 0 \le r \le a, \ \pi/2 \le \theta \le \pi\}$

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y \, dx \, dy = \int_0^a \int_{\pi/2}^{\pi} (r\cos\theta)^2 (r\sin\theta) \, r \, dr \, d\theta = \int_0^a r^4 dr \underbrace{\int_{\pi/2}^{\pi} \cos^2\theta \sin\theta \, d\theta}_{u = \cos\theta}$$

$$= \left[\frac{r^5}{5}\right]_0^a \left[\frac{-\cos^3\theta}{3}\right]_{\pi/2}^{\pi} = \frac{a^5}{5} \cdot \frac{1}{3} = \frac{1}{15} a^5$$

7)



Then, D is a Type I region, $D = \{(x,y) \mid 0 \le x \le 2, \ x \le y \le 6-2x\}$

Then, $\displaystyle m = \iint_D \rho(x,y) \, dA = \int_0^2 \int_x^{6-2x} x^2 \, dy \, dx = \int_0^2 \left[x^2 y\right]_x^{6-2x} dx = \int_0^2 6x^2 - 3x^3 \, dx = \left[2x^3 - \frac{3x^4}{4}\right]_0^2$

$= 16 - 12 = 4$.

Next,

$$M_y = \iint_D x\rho(x,y) \, dA = \int_0^2 \int_x^{6-2x} x \cdot x^2 \, dy \, dx = \int_0^2 x^3 y \Big]_{y=x}^{6-2x} dx = \int_0^2 6x^3 - 3x^4 \, dx = \left[\frac{3x^4}{2} - \frac{3x^5}{5}\right]_0^2$$

$$= 24 - \frac{96}{5} = \frac{24}{5}$$

$$M_x = \iint_D y\rho(x,y) \, dA = \int_0^2 \int_x^{6-2x} x^2 y \, dy \, dx = \frac{1}{2} \int_0^2 \left[x^2 y^2\right]_x^{6-2x} dx = \frac{1}{2} \int_0^2 36x^2 - 24x^3 + 4x^4 - x^4 \, dx$$

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$$= \frac{1}{2}\left[12x^3 - 6x^4 + \frac{3x^5}{5}\right]_0^2 = \frac{1}{2}\left[\frac{96}{5}\right] = \frac{48}{5}$$

Finally,

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{24/5}{4}, \frac{48/5}{4}\right) = \left(\frac{6}{5}, \frac{12}{5}\right)$$

8) $$f(x,y) = \begin{cases} 0.1e^{-(0.5x+0.2y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

a) $f(x,y) \geq 0$ since exponential function is always positive.

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^\infty \int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$

$$= 0.1 \int_0^\infty e^{-0.5x}\,dx \int_0^\infty e^{-0.2y}\,dy$$

$$= 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \cdot \lim_{s\to\infty} \int_0^s e^{-0.2y}\,dy$$

$$= 0.1 \lim_{t\to\infty}\left[\frac{e^{-0.5t}}{-0.5} - \frac{e^{-0.5(0)}}{-0.5}\right] \cdot \lim_{s\to\infty}\left[\frac{e^{-0.2s}}{-0.2} - \frac{e^{-0.2(0)}}{-0.2}\right] \quad \left(\text{Use, } \lim_{x\to\infty} e^{-cx} = 0 \ \forall c > 0\right)$$

$$= 0.1\left(\frac{1}{0.5}\right)\left(\frac{1}{0.2}\right) = 1$$

Hence f is a joint density function.

b)

i) $$P(Y \geq 1) = \int_1^\infty \int_0^\infty 0.1e^{-0.5x} e^{-0.2y}\,dx\,dy$$

$$= 0.1 \int_1^\infty e^{-0.2y}\,dy \cdot \int_0^\infty e^{-0.5x}\,dx = 0.1 \lim_{t\to\infty}\left[\frac{e^{-0.2y}}{-0.2}\right]_1^t \cdot \underbrace{\frac{1}{0.5}}_{\text{from part a}} = 0.1 \cdot \left(\frac{e^{-0.2}}{0.2}\right) \cdot 2 = e^{-0.2}$$

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ii) $P(X \le 2, Y \le 4) = \int_0^2 \int_0^4 0.1 e^{-(0.5x+0.2y)} \, dy \, dx = 0.1 \int_0^2 e^{-0.5x} dx \int_0^4 e^{-0.2y} dy$

$= 0.1 \left[\frac{e^{-0.5x}}{-0.5} \right]_0^2 \cdot \left[\frac{e^{-0.2y}}{-0.2} \right]_0^4$

$= 0.1 \left(\frac{e^{-1}}{-0.5} - \frac{1}{-0.5} \right) \left(\frac{e^{-0.8}}{-0.2} - \frac{1}{-0.2} \right)$

$= (1 - e^{-1})(1 - e^{-0.8})$

c) $E(X) = \int_0^\infty \int_0^\infty 0.1\, x e^{-(0.5x+0.2y)} \, dy \, dx$, $E(Y) = \int_0^\infty \int_0^\infty 0.1\, y e^{-(0.5x+0.2y)} \, dy \, dx$

9)

a) So $z = f(x,y)$ where $1 \le x^2 + y^2 \le 4$

$f_x = -2x, \; f_y = 2y$

$A(S) = \iint_D \sqrt{1 + (-2x)^2 + (2y)^2} \, dA$ where $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4\}$
$= \{(r,\theta) \mid 1 \le r \le 2, \; 0 \le \theta \le 2\pi\}$

$= \iint_D \sqrt{1 + 4(x^2+y^2)} \, dA$

$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} \, r \, dr \, d\theta$

$= \int_0^{2\pi} d\theta \underbrace{\int_1^2 r\sqrt{1+4r^2} \, dr}_{\text{Use } u = 1+4r^2} = 2\pi \left[\frac{1}{12} (1+4r^2)^{3/2} \right]_1^2$

$= \frac{\pi}{6} \left((17)^{3/2} - (5)^{3/2} \right).$

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b) Here, $z = \sqrt{4-x^2-y^2}$ [Don't need to consider $z = -\sqrt{4-x^2-y^2}$ as we are interested in the part that lies above the xy-plane]

and within the cylinder

$x^2+y^2 = 2x$

$\Rightarrow x^2-2x+y^2=0$

$\Rightarrow (x^2-2x+1)+y^2 = 1$

$\Rightarrow (x-1)^2+y^2 = 1$ (Circle of radius 1 centered at $(1,0)$)

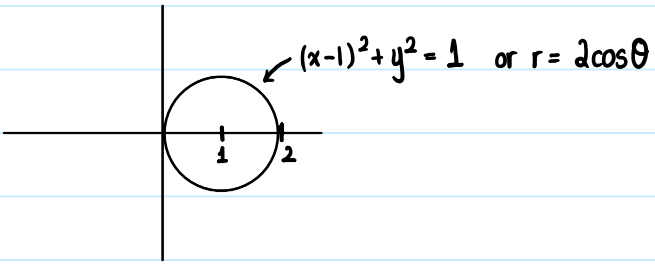
- $(x-1)^2+y^2 = 1$

$(r\cos\theta-1)^2+(r\sin\theta)^2 = 1$

$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$

$r^2 - 2r\cos\theta = 0$

$r = 2\cos\theta$



The disc can be expressed as

$$D = \{(r,\theta) \mid -\pi/2 \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le 2\cos\theta\}.$$

Then $f_x = \dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y = \dfrac{-y}{\sqrt{4-x^2-y^2}}$

$$A(S) = \iint_D \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

$$= \iint_D \sqrt{1+\frac{x^2+y^2}{4-x^2-y^2}}\, dA$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \sqrt{\frac{r^2}{4-r^2}+1}\; r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \sqrt{\frac{r^2+4-r^2}{4-r^2}}\; r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{2r}{\sqrt{4-r^2}}\, dr\, d\theta$$

Let $u = 4 - r^2$

$$\frac{du}{-2} = dr$$

$$= \int_{-\pi/2}^{\pi/2} \int_{r=0}^{r=2\cos\theta} \frac{-du}{u^{1/2}} \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[-2\sqrt{4 - r^2} \right]_0^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -2\sqrt{4 - 4\cos^2\theta} - (-2)\sqrt{4} \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4 - 2\sqrt{4(1-\cos^2\theta)} \, d\theta = 4 \int_{-\pi/2}^{\pi/2} 1 - \sqrt{1-\cos^2\theta} \, d\theta$$

$$= 8 \int_0^{\pi/2} 1 - \sin\theta \, d\theta = 8 \left[\theta + \cos\theta \right]_0^{\pi/2} = 8\left(\frac{\pi}{2} - 1\right) = 4(\pi - 2)$$

10)

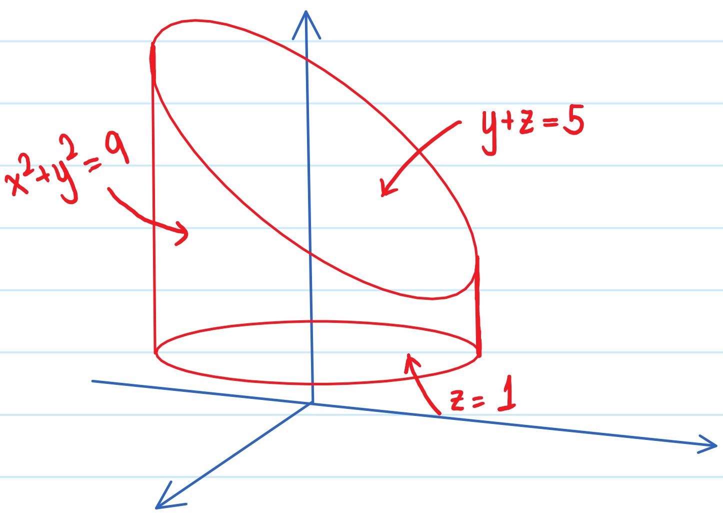


Diagram of Region E. (Not necessary to solve the problem).

First note that the region in question is bounded by planes $z = 5 - y$ and $z = 1$ (So region is Type 1)

Therefore , $V(E) = \iiint\limits_E 1\, dV = \iint\limits_D \int_1^{5-y} 1\, dz\, dA = \iint\limits_D (5-y-1)\, dA = \iint\limits_D 4-y\, dA$

Next we Know that the region $D = \{(x,y) \mid x^2+y^2 \le 9\}$ or in polar coordinates
$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3\}$

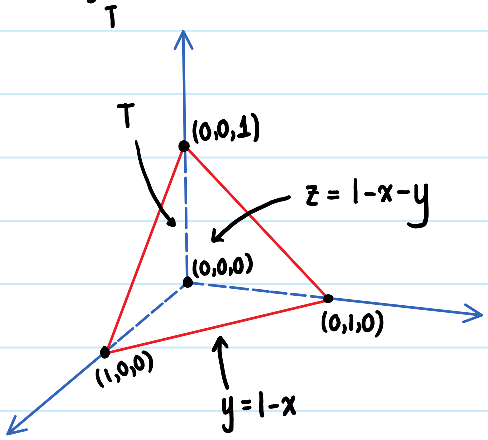
Then,

$$\iint\limits_D 4-y\, dA = \int_0^{2\pi}\int_0^{3} (4 - r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{r} 4r - r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{1}{3}r^3\sin\theta\right]_0^3 d\theta = \int_0^{2\pi} 18 - 9\sin\theta = 18\theta + 9\cos\theta\Big]_0^{2\pi} = 36\pi$$

11)

a) $\iiint\limits_T x^2\, dV$



The front face of the tetrahedron lies on the plane
$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1 \Rightarrow x+y+z = 1$

In general, a plane w/ x intercept a , y-int b and z-int c is given by the formula $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Then we see that T can be expressed as a type 1 region $\{(x,y) \in D \mid 0 \le z \le 1-x-y\}$

To determine region D , we find the projection of $z = 1-x-y$ on the xy-axis, which gives us $y = 1-x$, and from the diagram we see that $0 \le x \le 1$.

Therefore, $E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$

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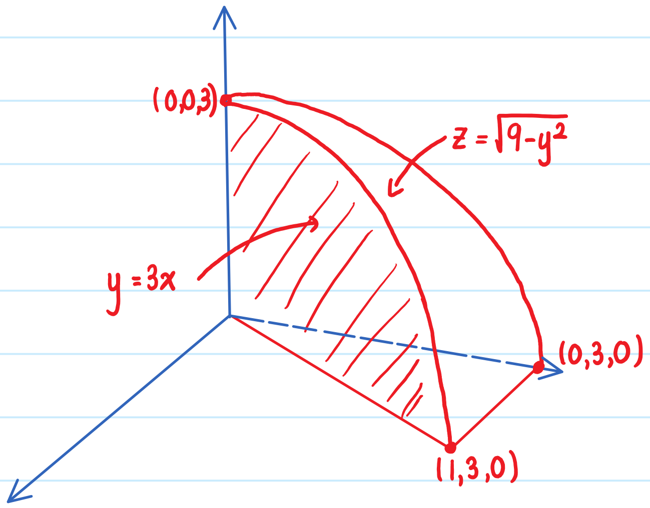
Therefore, $\iiint_T x^2\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\,dz\,dy\,dx$

$$= \int_0^1\int_0^{1-x} x^2(1-x-y)\,dy\,dx = \int_0^1\int_0^{1-x} x^2-x^3-x^2y\,dy\,dx = \int_0^1\left[x^2y-x^3y-\frac{x^2y^2}{2}\right]_{y=0}^{y=1-x}dx$$

$$= \int_0^1 (1-x)\left[x^2-x^3-\frac{1}{2}x^2(1-x)\right]dx$$

$$= \int_0^1 \frac{1}{2}x^4-x^3+\frac{1}{2}x^2\,dx = \frac{1}{10}-\frac{1}{4}+\frac{1}{6} = \frac{1}{60}$$

b)



Then $\iiint_E z\,dV = \int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1\int_{3x}^3 \frac{1}{2}(9-y^2)\,dy\,dx$

$$= \int_0^1\left[\frac{9}{2}y-\frac{y^3}{6}\right]_{3x}^{3}dx = \int_0^1 \frac{27}{2}-\frac{9}{2}-\left(\frac{27}{2}x-\frac{9}{2}x^3\right)dx$$

$$= \int_0^1 9-\frac{27}{2}x+\frac{9}{2}x^3\,dx = \left[9x-\frac{27x^2}{4}+\frac{9x^4}{8}\right]_0^1 = \frac{27}{8}$$

12) $E = \{(x,y,z) \mid 0 \le x \le 1-z,\ 0 \le z \le 1-y^2,\ -1 \le y \le 1\}$

Then, $m = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4\,dx\,dz\,dy = 4\int_{0}^{1}\int_{0}^{1-y^2}(1-z)\,dz\,dy = 4\int_{-1}^{1}\left[z-\frac{1}{2}z^2\right]_{z=0}^{1-y^2} = 2\int_{-1}^{1}(1-y^4)\,dy = \frac{16}{5}$

- $M_{yz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^{1}\int_{0}^{1-y^2}(1-z)^2\,dy\,dx = \frac{2}{3}\int_{-1}^{1} 1-y^6\,dy = \frac{24}{21}$

Integral of odd func over symmetric interval

$M_{xz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4y\,dx\,dy\,dz = \int_{-1}^{1}\int_{0}^{1-y^2} 4y-4yz\,dz\,dy = \int_{-1}^{1}\left[4yz-2yz^2\right]_{z=0}^{1-y^2}dy = \int_{-1}^{1} 2y-2y^5\,dy = 0$

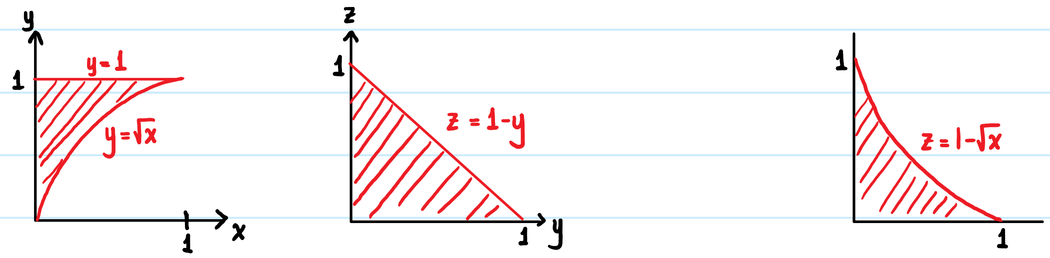
$M_{xy} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_{0}^{1-y^2} 4z-4z^2\,dz\,dy = 2\int_{-1}^{1}(1-y^2)^2-\frac{2}{3}(1-y^2)^3\,dy$

$= 2\int_{-1}^{1}\frac{1}{3}-y^4+\frac{2}{3}y^6\,dy = \frac{96}{105}$

Then, $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

$= \left(\frac{24/21}{16/5}, \frac{0}{16/5}, \frac{32/35}{16/5}\right) = \left(\frac{5}{14}, 0, \frac{2}{7}\right).$

13) The projection of the region of integration E on the xy, xz and yz planes are



Then,

① $\int_{0}^{1}\int_{\sqrt{x}}^{1}\int_{0}^{1-y} f(x,y,z)\,dz\,dy\,dx$ = ② $\int_{0}^{1}\int_{0}^{y^2}\int_{0}^{1-y} f(x,y,z)\,dz\,dx\,dy$ = ③ $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{y^2} f(x,y,z)\,dx\,dy\,dz$

= ④ $\int_{0}^{1}\int_{0}^{1-y}\int_{0}^{y^2} f(x,y,z)\,dx\,dz\,dy$ = ⑤ $\int_{0}^{1}\int_{0}^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-x} f(x,y,z)\,dy\,dz\,dx$ = ⑥ $\int_{0}^{1}\int_{0}^{(1-x)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$